SECTION 6.3: VOLUME BY SLICING

THEOREM: Suppose $A(t)$, $a \le t \le b$ describes the area of cross-sections of a solid obtained by slicing the solid perpendicular to the t-axis. The volume of the solid is given by:

$$V = \int_a^b A(t)\,dt$$

EXAMPLE 1: Let R be the region bounded by the graphs of $y = x^2$, $y = \sqrt{x}$.

Find the volume of the solid whose cross-sections perpendicular to the base and parallel to the y-axis are:

1. Squares

$$\text{Ans: } V = \int_0^1 \left(\sqrt{x} - x^2\right)^2\,dx = \ldots = \frac{9}{70}\text{ units}^3$$

2. Equilateral Triangles

$$\text{Ans: } V = \int_0^1 \frac{\sqrt{3}}{4}\left(\sqrt{x} - x^2\right)^2\,dx = \ldots = \frac{9\sqrt{3}}{280}\text{ units}^3$$

3. Semicircles

$$\text{Ans: } V = \int_0^1 \frac{\pi}{2}\left(\frac{\sqrt{x} - x^2}{2}\right)^2\,dx = \ldots = \frac{9\pi}{560}\text{ units}^3$$

EXAMPLE 2: Find the volume of the solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the x-axis are equilateral triangles.

$$\text{Ans: } V = \int_{-5}^5 \frac{\sqrt{3}}{4}\left(2\sqrt{25 - y^2}\right)^2\,dy = 2\sqrt{3}\int_0^5 \left(25 - y^2\right)\,dy = \ldots = \frac{500\sqrt{3}}{3}\text{ units}^3$$

SOLIDS OF REVOLUTION: If a region in the plane is rotated about an axis which does not intersect the interior of the region, the resulting solid is called a **solid of revolution**. Note that a solid of revolution may be thought of as an aggregate of infinitely many infinitesimally thin disks stacked together.

To find the volume of a solid of revolution, we chop up the axis of rotation and use the fact that the volume of solid can be approximated by summing the volumes of 'disks' with volume:

$$V_{\text{disk}} = \pi\,(\text{radius})^2\,(\text{thickness})$$

Here, the 'thickness' of the disk is dx or dy (whichever axis we chopped up) and the 'radius' is the distance from the curve to the axis of rotation. Finding the volume in this way is the so-called **disk method**.

EXAMPLE 3: Find the volume of the solid obtained by revolving the given region about the stated axis:

1. The region bounded by $y = x^2$ and $y = 0$ from $x = 0$ to $x = 2$ revolved about the x-axis.

$$\text{Ans: } V = \int_0^2 \pi\left(x^2\right)^2\,dx = \ldots = \frac{32\pi}{5}\text{ units}^3$$

2. The region bounded by $y = x^2$ and $y = -1$ from $x = 0$ to $x = 2$ revolved about the line $y = -1$.

$$\text{Ans: } V = \int_0^2 \pi\left(x^2 + 1\right)^2\,dx = \ldots = \frac{206\pi}{15}\text{ units}^3$$

EXAMPLE 4: Let R be the unbounded region between the graph of $y = \dfrac{1}{x}$ and the x-axis for $x \geq 1$.

Gabriel's Horn is the solid of revolution obtained by rotating R about the x-axis.

Write and evaluate an improper integral which would compute the volume of Gabriel's Horn.

$$\text{Ans: } V = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \ldots = \pi \text{ units}^3$$

Suppose we wish to use the disk method to find the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = \sqrt{8x}$ about the x-axis. Sketching a representative rectangle reveals we have a 'gap' between the bottom of the region and the axis of rotation. The object which forms as a result of this rotation is called a **washer** - or, as I like to call it, a 'disk with a hole.' We use the principle: 'volume = whole − hole' to get:

$$V_{\text{washer}} = \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] (\text{thickness})$$

EXAMPLE 5: Let R be the region bounded by $y = x^2$ and $y = \sqrt{8x}$.

Find the volume of the solid obtained by revolving R about:

1. the x-axis.

$$\text{Ans: } V = \int_0^2 \pi \left[\left(\sqrt{8x}\right)^2 - \left(x^2\right)^2\right] dx = \ldots = \frac{48\pi}{5} \text{ units}^3$$

2. the line $y = 5$.

$$\text{Ans: } V = \int_0^2 \pi \left[\left(5 - x^2\right)^2 - \left(5 - \sqrt{8x}\right)^2\right] dx = \ldots = \frac{256\pi}{15} \text{ units}^3$$

3. the y-axis.

$$\text{Ans: } V = \int_0^4 \pi \left[(\sqrt{y})^2 - \left(\frac{y^2}{8}\right)^2\right] dy = \ldots = \frac{24\pi}{5} \text{ units}^3$$

HOMEWORK: Section 6.3: 11-55 odd